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The limits for  $u$  are 0 and  $\pi/2$ , and as the equation of the given plane is  $\tan t = \tan \alpha \sin u$ , the limits for  $t$  will be  $\tan^{-1}(\tan \alpha \sin u)$  and  $\pi/2$ .

Therefore

$$S = \frac{\sqrt{a^2 + b^2}}{2} \int_0^{\pi/2} du \int_{\tan^{-1}(\tan \alpha \sin u)}^{\pi/2} (a + a \cos 2t + b \sin 2t) dt.$$

**2772 [1919, 171]. Proposed by HARRY LANGMAN, New York City.**

Given  $1 = (-\frac{1}{2} + x)^r = (-\frac{1}{2} - x)^r$ , where  $r$  is integral. Prove that  $r$  is a multiple of 3.

In general, if

$$1 = \left[ \cos \frac{2\pi}{m} + x \right]^r = \left[ \cos \frac{2\pi}{m} - x \right]^r,$$

where  $r$  and  $m$  are integers, prove that  $r$  is a multiple of  $m$ .<sup>1</sup>

### SOLUTION BY THE PROPOSER.

The general case only will be considered, and the trivial case  $r = 0$  excluded.

Put

$$(1) \quad a = \cos \frac{2\pi}{m} + x \quad \text{and} \quad b = \cos \frac{2\pi}{m} - x.$$

Then

$$(2) \quad a + b = 2 \cos \frac{2\pi}{m}.$$

We may write

$$(3) \quad 1 = \cos 2k\pi + i \sin 2k\pi.$$

Hence,

$$(4) \quad a = \cos \frac{2k\pi}{r} + i \sin \frac{2k\pi}{r} \quad \text{and} \quad b = \cos \frac{2k'\pi}{r} + i \sin \frac{2k'\pi}{r}, \quad 0 < k, k' < r + 1.$$

From (2),

$$\cos \frac{2k\pi}{r} + \cos \frac{2k'\pi}{r} + i \left( \sin \frac{2k\pi}{r} + \sin \frac{2k'\pi}{r} \right) = 2 \cos \frac{2\pi}{m},$$

from which

$$(5) \quad \cos(k + k') \frac{\pi}{r} \cdot \cos(k - k') \frac{\pi}{r} = \cos \frac{2\pi}{m} \quad \text{and} \quad \sin(k + k') \frac{\pi}{r} \cdot \cos(k - k') \frac{\pi}{r} = 0.$$

Hence, we must have

$$(6) \quad \sin(k + k') \frac{\pi}{r} = 0.$$

From the problem, if  $m \neq 1$ , we must have  $a \neq b$ . Hence, in (4)  $k \neq k'$ . Hence, we must have  $k + k' < 2r$ , that is,  $(k + k')/r < 2$ . But, from (6),  $(k + k')/r$  must be integral. Hence, since  $k + k' > 0$ ,

$$(7) \quad k + k' = r \quad \text{and} \quad k - k' = 2k - r.$$

From the first equation of (5), we then obtain (8)  $\cos \frac{2\pi}{m} = -\cos \frac{2k - r}{r} \pi$ .

Now  $\frac{2}{m} \leq 1$  and  $\left| \frac{2k - r}{r} \right| \leq 1$ . Therefore, from (8),

$$\left| \frac{2k - r}{r} \right| = 1 - \frac{2}{m}.$$

Hence,  $\frac{2k}{r} - 1 = 1 - \frac{2}{m}$  or  $1 - \frac{2k}{r} = 1 - \frac{2}{m}$ , from which  $k = r - \frac{r}{m}$  or  $k = \frac{r}{m}$ . From (7),

$k' = \frac{r}{m}$  or  $k' = r - \frac{r}{m}$ . Since  $k$  and  $k'$  are integral, we must have  $\frac{r}{m}$  integral. Hence,  $r$  is a multiple of  $m$ .

<sup>1</sup> If  $m = 4$  we may take  $x = 1$ ,  $r = 2$ ; in this case  $r$  is not a multiple of  $m$ . Therefore the theorem of the question is not true for this case.—EDITORS.

Also solved by C. A. BARNHART, H. HALPERIN, H. L. OLSON, and A. PELLETIER.

**2773 [1919, 212]. Proposed by JOSEPH ROSENBAUM, Milford, Conn.**

Point out the fallacy in the proof of the following problem:

In the triangle  $A_1B_1C_1$  let  $m$  be a point such that the sum of the distances from it to the sides is a maximum; also let  $A_2B_2C_2$  be a triangle formed by drawing lines through the vertices  $A_1$ ,  $B_1$ , and  $C_1$  parallel to their opposite sides. Then the sum of the distances from  $m$  to the sides of the triangle  $A_2B_2C_2$  is a minimum.

*Proof.*—Because the sides of the two triangles are parallel in pairs, the sum of the distances from a variable point  $P$  in triangle  $A_1B_1C_1$  to the six sides of the two triangles is constant. Now by hypothesis  $M$  is a point for which one part of this constant sum is a maximum, and hence it follows that the other part is a minimum.

SOLUTION BY H. L. OLSON, University of Wisconsin.

This proof is correct, with the understanding that if a point  $P$  is on the opposite side of  $BC$ , for example, to the vertex  $A$ , the distance to the side  $BC$  is to be regarded as negative. It is easy to see, however, that the point  $M$  does not exist, and that the proposition is therefore vacuous. Represent the perpendicular distances from  $P$  to the sides  $BC$ ,  $AC$ , and  $AB$  by  $\alpha$ ,  $\beta$ , and  $\gamma$  respectively. If we denote by  $\Delta$  the area of the triangle  $ABC$ , we are to minimize the function  $\alpha + \beta + \gamma$ , subject to the condition  $a\alpha + b\beta + c\gamma = 2\Delta$ . ( $a$ ,  $b$ , and  $c$  represent, as is customary, the sides  $BC$ ,  $AC$ , and  $AB$ , respectively.) Eliminating  $\gamma$ , we have, as the function to be minimized,

$$\left(1 - \frac{a}{c}\right)\alpha + \left(1 - \frac{b}{c}\right)\beta + \frac{2\Delta}{c}.$$

Hence, the derivatives,  $\left(1 - \frac{a}{c}\right)$ , and  $\left(1 - \frac{b}{c}\right)$ , of this function with respect to  $\alpha$  and  $\beta$

must vanish; but for the general triangle they do not vanish and hence  $M$  does not exist. If, however,  $a = b = c$ , the sum of the distances is the constant  $2\Delta/c$ ; likewise the sum of the distances for the corresponding triangle  $A_2B_2C_2$  is constant.

Also solved by A. PELLETIER and A. L. WECHSLER.

**2774 [1919, 212]. Proposed by FRANK IRWIN, University of California.**

Evaluate the circulants

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{vmatrix},$$

where, in the latter,  $a_1, a_2, \dots, a_n$  form an arithmetical progression.

I. SOLUTION BY P. J. DA CUNHA, University of Lisbon, Portugal.

Denote the first of these circulants by  $\Delta$  and the second by  $\Delta^a$ . Let

$$s_n = \frac{1+n}{2}n$$

be the sum of the first  $n$  positive integers. Add to the elements of the last line of  $\Delta$  the sum of the corresponding elements of all the preceding lines. We obtain a determinant which we can write as the product

$$\Delta = s_n \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ n & 1 & 2 & \cdots & n-3 & n-2 & n-1 \\ n-1 & n & 1 & \cdots & n-4 & n-3 & n-2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & 5 & 6 & \cdots & 1 & 2 & 3 \\ 3 & 4 & 5 & \cdots & n & 1 & 2 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix}.$$